Random walks on site disordered lattices

1985 J. Phys. A: Math. Gen. 18 L531

(http://iopscience.iop.org/0305-4470/18/9/008)

J. Phys. A: Math. Gen. 18 (1985) L531-L538. Printed in Great Britain

LETTER TO THE EDITOR

Random walks on site disordered lattices

Jonathan Machta

Department of Physics and Astronomy, University of Massachusetts, Amherst, MA 01003, USA

Received 4 April 1985

Abstract. Random walk models with disordered waiting times (site disorder) are analysed using a real space renormalisation group method. Anomalous diffusion is found in all dimensions whenever the first moment of the distribution of waiting times diverges. Above two dimensions the fixed points are described by continuous time random walks with non-analytic waiting time functions. Below two dimensions the fixed points are themselves disordered random walks where the disorder is characterised by a stable probability distribution. A fractal dimension is defined for site disordered lattices and, below two dimensions, a hyperscaling law is found which relates the fractal dimension, the Euclidean dimension, and the exponent describing the displacement of the walker.

Diffusion in disordered media has been the subject of considerable recent interest. The anomalous transport properties which result from the presence of disorder have been studied with a variety of random walk models including walks on lattices with random biases (Sinai 1982, Luck 1983, Fisher 1984), walks on one-dimensional lattices with random bonds or random sites (Alexander *et al* 1981, Stephen and Kariotis 1982, Nieuwenhuizen and Ernst 1985) and walks on percolation clusters (de Gennes 1976). Walks on ordered lattices which display anomalous transport have also been studied. These include walks on self similar or fractal lattices (Gefen *et al* 1981), continuous time random walks (Schlesinger 1974) and walks with random step lengths (Gillis and Weiss 1970). For ordered models there are exact results in all dimensions but for disordered models exact results are difficult to obtain except in one dimension.

This letter presents an investigation of random walks on lattices with site disorder. Site disorder is defined to mean that the expected waiting time at each lattice site is a quenched random variable. The effect of site disorder is studied using a real space renormalisation group procedure closely related to the method introduced by Machta (1981, 1983). We find that anomalous diffusion ($r \sim t^{\nu}$; $\nu < \frac{1}{2}$) occurs whenever the distribution of expected waiting times has no first moment. The exponents which we obtain agree with the heuristic treatment of the problem given by Alexander (1981) who showed that $d = 2$ is the upper critical dimension for the problem. Above two dimensions the fixed point of the renormalisation group flow corresponds to a continuous time random walk with a non-analytic waiting time function of the kind studied by Schlesinger (1974). The continuous time random walk approximation discussed by Scher and Lax (1973), Scher and Montroll (1975) and Klafter and Silbey (1980) corresponds to the mean field theory for the problem. Below two dimensions, for sufficiently strong disorder, the fixed points are themselves disordered random walks where the disorder is described by a stable probability distribution. We obtain a

0305-4470/85/090531+08$02.25

hyperscaling relation, valid for $d<2$, which agrees with the exact one-dimensional results of Alexander *et al* (1981) and Stephen and Kariotis (1982).

The concept of fractal dimension (see Mandelbrot 1982) plays a key role in the description of scale invariant structures such as percolation clusters or Sierpinski gaskets. We show that site disordered lattices enjoy a statistical scale invariance which can be quantified by a fractal dimension, d_f. The fractal dimension enters the hyperscaling relation obtained from the renormalisation group analysis and relates it to ν and d.

We consider random walks on a d-dimensional hypercubic lattice with lattice constant a. At each lattice site j there is a quenched random variable τ_j which is the expected waiting time for jumping from the site. The walker jumps with equal probability to any of the 2D nearest neighbours. The τ_j are independent, identically distributed (IID) positive random variables with distribution $F(\tau)$.

The random walk may be described either by a master equation or as a continuous time random walk (CTRW). The master equation takes the form

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t}=\sum_j\,(p_j(t)/\tau_j-p_i(t)/\tau_i) \tag{1}$$

where the sum extends over nearest neighbours of i and $p_i(t)$ is the probability at time t that the walker is at site i. Note that, in equilibrium, p_i is proportional to τ_i. In the CTRW description a waiting time function, $\tilde{\psi}_i(t)$ is assigned to each site such that $\tilde{\psi}_i(t)\,\mathrm{d}t$ is the probability that the walker leaves site i in the time interval t to $t+\mathrm{d}t$ after arrival. As is well known (see, e.g., Klafter and Silbey 1980) a CTRW is equivalent to a generalised master equation and the special case of the Markovian master equation (1) is described by exponential waiting time functions,

$$\tilde{\psi}_i(t)=(1/\tau_i)\,\mathrm{e}^{-t/\tau_i}. \tag{2}$$

We consider three classes of distributions for τ_i. Class a or weak disorder contains all distributions with finite means, μ,

$$\mu=\int x\,\mathrm{d}F(x). \tag{3}$$

Haus *et al* (1982) have obtained an exact solution for the weak disorder problem. Classes b and c, together called strong disorder, contain all distributions without first moments but with regularly varying tails

$$(1-F(x))\sim x^{-\alpha} \tag{4}$$

where $\alpha=1$ for class b and $0<\alpha<1$ for class c. The notation $f(x)\sim x^{\alpha}$ means $\lim_{x\to\infty}f(\lambda x)/f(x)=\lambda^{\alpha}$ for all $\lambda>0$. This classification scheme agrees (except for a redefinition of α) with the one introduced by Alexander *et al* (1981) and also incorporates distributions whose tails are not pure power laws but have, for example, logarithmic corrections. As α becomes smaller, the system becomes more strongly disordered since the likelihood of anomalously long waiting times increases.

A variety of physical systems are isomorphic to the random walk described by the master equation (1). Among these are harmonic lattices with random masses, tight-binding models with diagonal disorder, and electrical networks of resistors and random capacitors. In the electrical network interpretation each lattice point has a capacitor connected to ground, nearest-neighbour capacitors are connected by unit resistors and p_i is the charge on the capacitor at site i.

The primary objective of this work is to obtain the exponents ν and d_s describing the asymptotic behaviour of the root-mean-squared displacement of the walk, $R(t)$ and the probability of being at the origin $P_0(t)$. These quantities are defined by

$$R(t) \equiv \left\langle \left(\sum_j r_j^2 p_j(t) \right)^{1/2} \right\rangle \sim t^{\nu} \tag{5}$$

and

$$P_0(t) \equiv \langle p_0(t) \rangle \sim t^{-d_s/2} \tag{6}$$

where r_j is the position of site j, and the brackets indicate an average over the configurations of the disordered lattice. The initial condition in both cases is $p_j(0) = \delta_{j0}$. The exponent d_s is often referred to as the spectral dimension because of its relation to the density of states of the equivalent harmonic lattice.

In order to determine the asymptotic properties of the walk we use a real space RG method similar to the one used by Machta (1981, 1983). The first step in the RG transformation is to replace (hypercubic) blocks of 2^d sites by a single site on a hypercubic lattice with lattice constant $2a$. The random walk on the new lattice is described by a set of block waiting time functions $\tilde{\psi}^{\mathrm{b}}(t)$ chosen to preserve the long time and large distance properties of the walk. The block waiting time functions are obtained from the waiting time functions in the block by averaging over paths for leaving the block. This is most easily accomplished using Laplace transformed waiting time functions,

$$\psi(z) = \int_0^{\infty} \mathrm{d}t \, \mathrm{e}^{-2t} \tilde{\psi}(t). \tag{7}$$

A path $\Gamma = (j_1, j_2, \ldots, j_{n(\Gamma)})$ is a sequence of $n(\Gamma)$ sites all within a block. The walker arrives at site j_1, visits the sequence of sites, Γ and then leaves the block from site $j_{n(\Gamma)}$. The waiting time function, $\tilde{\psi}_\Gamma(t)$, for making the sequence of jumps Γ is the convolution of waiting time functions along the path so that its Laplace transform is given by

$$\psi_\Gamma(z) = \prod_{m=1}^{n(\Gamma)} \psi_{j_m}(z). \tag{8}$$

We define the block waiting time function, $\psi_i^{\mathrm{b}}(z)$, as

$$\psi_i^{\mathrm{b}}(z) = \sum_\Gamma w(\Gamma) \psi_\Gamma(z) \tag{9}$$

where the sum is over all paths for leaving the block i and $w(\Gamma)$ is the weight assigned to the path. For the analysis which follows it is unnecessary to define the weight function explicitly. We require only that it have the following properties.

(1) Normalisation

$$\sum_\Gamma w(\Gamma) = 1. \tag{10}$$

(2) The mean number of sites visisted per visit to a block is 4,

$$\sum_\Gamma n(\Gamma) w(\Gamma) = 4. \tag{11}$$

(3) Each site in the block is visited with equal weight.

For the case of weakly disordered (and ordinary) random walks these properties are sufficient to show that the diffusion coefficient, $D = a^2/2d\mu$ (Haus *et al* 1982) is an invariant of the transformation. Site disorder does not affect the weighting of paths contributing to the Green function. Thus we hypothesise that the asymptotic properties of strongly disordered walks will also be invariants of the transformation.

It is the basic assumption of this work that a random walk given by (8) and (9) with a weight function satisfying properties (1)-(3) has the same long time and large distance properties as the original random walk.

The next step in the RG procedure is to rescale space and time so that the RG flow leads to a non-trivial fixed point. By non-trivial we mean that $0 < R(t) < \infty$ for $0 < t < \infty$. In order that the lattice constant be invariant under the RG transformation, the space rescaling factor must be chosen to be two. The time rescaling factor, λ, enters the definition of the renormalised waiting time functions, $\tilde{\psi}'_i(t)$ via

$$\tilde{\psi}'_i(t) = \tilde{\psi}^{\mathrm{b}}_i(t\lambda). \tag{12}$$

The choice of time rescaling factor at each RG transformation depends on the set of waiting time functions and is dictated by the requirement that the recursion relations defined by equations (8)-(12) flow to a non-trivial fixed point.

The fixed points of the RG flow are more easily understood in the small z limit by considering recursion relations on the set of functions $T_j(z)$ defined by

$$T_j(z) \equiv \frac{-1}{\psi_j(z)} \frac{\mathrm{d}}{\mathrm{d}z} \psi_j(z). \tag{13}$$

Note that $T_j(0)$ is the expected waiting time at site j. The recursion relations for the functions $T_j(z)$ are determined from (8), (9), (12) and (13) and take the form

$$T'_i(z) = (1/\lambda) \sum_{\Gamma} w(\Gamma)\psi_\Gamma(z/\lambda)\left(\sum_{m=1}^{n(\Gamma)} T_{j_m}(z/\lambda)\right)\left(\sum_{\Gamma} w(\Gamma)\psi_\Gamma(z/\lambda)\right)^{-1}. \tag{14}$$

For small z, $\psi(z)$ can be replaced by 1 and, using properties (1) through (3) of the weight function, we find that the recursion relations take the simple form

$$T'(z) = (4/\lambda)2^{-d} \sum_{j\in C} T_j(z/\lambda) \tag{15}$$

where C is the set of 2^d sites in the block and, to simplify the notation, we have omitted the lattice index of the block. After l iterations of the recursion relations, $T^{(l)}(z)$ is a weighted sum of 2^{dl} random functions (see (2)),

$$T^{(l)}(z) = (4^l/Q(l))2^{-dl} \sum_{j\in C(l)} \frac{\tau_j}{(z\tau_j/Q(l)+1)} \tag{16}$$

where $C(l)$ is the set of 2^{dl} sites in the original lattice which are in the block and $Q(l)$ is the product of the time rescaling factors from the first l transformations,

$$Q(l) = \prod_{m=1}^{l} \lambda^{(m)}. \tag{17}$$

Our task is to choose a sequence of time rescaling factors so that, as $l \to \infty$, the set of functions $\{T_j^{(l)}\}$ goes to a non-trivial fixed point. To understand the behaviour of the sum (16) for large l we make use of theorem characterising the sum and maximum element of a set of IID random variables (see, e.g., Feller 1971).

Suppose $\{X_1, X_2, \ldots, X_N\}$ is a set of N IID positve random variables, $\bar{X}(N)$ is the largest element of the set and $Y(N)$ is a sum over the set,

$$Y(N) = \sum_{m=1}^{N} X_m. \tag{18}$$

For a class a distributions $\bar{X}(N)/Y(N) \to 0$ with probability 1 so that $\bar{X}(N)$ makes an infinitesimal contribution to the sum as $N \to \infty$. On the other hand, for classes b and c, the distribution for $Y(N)/\bar{X}(N)$ is non-degenerate so that the largest summand is (almost surely) a finite fraction of the whole.

The sum itself diverges but with proper choice of norming constants, $A(N)$, the distribution for the random variable $Y(N)/A(N)$ approaches a limit distribution, F^*. We shall see that these stable distributions play the role of non-trivial fixed points for $d < 2$. The proper choice of norming constants are given as follows:

Class a. If μ is finite the law of large numbers shows that, with the choice

$$A(N) = N \tag{19}$$

the limit distribution F^* is concentrated at μ.

Class b. If F is of the form given in (4) with $\alpha = 1$ a modified form of the law of large numbers asserts that there is a choice

$$A(N) \sim N \tag{20}$$

such that the limit distribution is concentrated at unity.

Class c. If F is of the form given in (4) with $0 < \alpha < 1$ then there is a sequence of norming constants

$$A(N) \sim N^{1/\alpha} \tag{21}$$

such that the limit distribution is a stable distribution with characteristic exponent α. For classes b and c more explicit prescriptions for $A(N)$ can be found in Feller (1971) from which it can be seen that the asymptotic behaviour of $A(N)$ is typically not that of a pure power law in N. Roughly speaking then, for classes b and c both $\bar{X}(N)$ and $Y(N)$ typically grow like $N^{1/\alpha}$.

We first consider weak disorder (class a). The law of large numbers ensures that $T^{(l)}(0) \to T^*(0) = \mu$ so long as $\lambda = 4$. Let $\bar{\gamma}(l)$ be the maximum of the set of expected waiting times in the block $C(l)$. Since $\bar{\gamma}(l)/4^l \to 0$, the choice $\lambda = 4$ eliminates the z dependence from the sum (16) and the fixed point waiting time function, $\psi^*(z)$, derived from the full recursion relations (8)-(12) is expected to be well behaved. The fixed point is thus an ordered random walk where the mean waiting time at each site is μ.

For classes b and c the behaviour of the sum (16) is more subtle. First suppose that $\bar{\gamma}(l)/Q(l) \to \infty$. Then for non-zero z and large l the sum can be replaced by an integral. Taking the limit $l \to \infty$ and using (4) to scale out the dependence of the integral on z and we obtain the fixed point function $T^*(z)$,

$$T^*(z) = z^{\alpha-1}\left(\lim_{l\to\infty} 4^l \int \mathrm{d}F(yQ(l))\frac{y}{y+1}\right). \tag{22}$$

Thus $T^{(l)}(z)$ approaches a non-trivial function if and only if the limit in the brackets in (22) is a positive constant. The required asymptotic behaviour for product of the time rescaling factors is then $Q(l) \sim 4^{l/\alpha}$. Since $\bar{\gamma}(l)$ grows like $2^{dl/\alpha}$ the premise that $\bar{\gamma}(l)/Q(l) \to \infty$ holds for $d > 2$ and fails for $d < 2$. Thus, for $d > 2$, the fixed point is a CTRW with a single waiting time function, $\psi^*(z)$ at each lattice site. For class c this waiting time function is non-analytic at $z=0$ and behaves like

$$(1-\psi^*(z)) \sim z^{\alpha}. \tag{23}$$

This asymptotic behaviour is precisely the one predicted by the CTRW approximation scheme (see, e.g., Klafter and Silbey 1980) and leads to anomalous diffusion with $\nu=\alpha/2$.

Next suppose that $\bar{\gamma}(l)/Q(l) \to 0$, then (3.10) reduces to a sum of IID variables and $T^{(l)}$ will be a random variable described by a non-trivial limit distribution if and only if $Q(l)$ is chosen so that (20) or (21) is satisfied. Comparing (16), (18), (20) and (21) we find that the required asymptotic behaviour for the time rescaling factors is $Q(l) \sim 2^{2+d/\alpha-d}$. The ratio $\bar{\gamma}(l)/Q(l)$ then grows like $2^{(d-2)/\alpha}$ so that the premise is satisfied for $d<2$ and fails for $d>2$. For class b the fixed point is an ordered CTRW with a finite mean waiting time whereas for class c the fixed point is a disordered CTRW where the waiting time at each site is chosen from a stable distribution with characteristic exponent α.

The main result of this section is that, to obtain a non-trivial fixed point, the asymptotic behaviour of the time rescaling factors must be

$$Q(l) \sim \begin{cases} 4^{l} & \text{classes a and b} \\ 4^{l/\alpha} & d>2, \text{ class c} \\ 2^{2+d/\alpha-d} & d<2, \text{ class c.} \end{cases} \tag{24}$$

The exponent ν can be ascertained from the asymptotic behaviour of the time rescaling factors as follows. According to our basic assumption we may calculate the asymptotic properties of the walk using either the original waiting time functions or the renormalised waiting time functions. Thus, taking into account the rescaling of space and time, the RMS displacement after l iterations of the RG transformation, $R^{(l)}(t)$, is related to $R(t)$ via the asymptotic equality

$$R(t) \approx 2^{l} R^{(l)}(t/Q(l)). \tag{25}$$

The notation $f(x) \approx g(x)$ means $\lim_{x\to\infty} f(x)/g(x)=1$. We now fix the argument of $R^{(l)}$ in (5.1) by choosing l so that $t/Q(l)=t_0$ where t_0 is some arbitrary large constant. As t increases so does l so that for large t we are close to the fixed point and can replace $R^{(l)}(t_0)$ by its value at the fixed point, $R^*(t_0)$. The large l behaviour of $Q(l)$ given in (24) determines ν through the asymptotic relation $Q(l)=(t/t_0) \sim 2^{l/\nu}$ and we obtain,

$$\nu = \begin{cases} \frac{1}{2} & \text{classes a and b} \\ \alpha/2 & d>2, \text{ class c} \\ [2+d/\alpha-d]^{-1} & d<2, \text{ class c.} \end{cases} \tag{26}$$

The asymptotic behaviour of the probability of being at the origin can be obtained using similar arguments. The renormalised probability of being at the origin is the sum of the probabilities of being at the 2^d sites in the block containing the origin. Thus

$$p_0'(t/\lambda) = \sum_{j\in c} p_j(t) \tag{27}$$

where the sum is over the sites making up the block at the origin. Since $\langle p_i(t)\rangle \approx \langle p_0(t)\rangle$ we have

$$p_0(t) \approx 2^{-dl} p_0^{(l)}(t/Q(l)) \tag{28}$$

comparing (28) and (25) we see that

$$d_s = 2d\nu. \qquad (29)$$

The present calculation has determined only the leading exponents and does not distinguish between pure power law behaviour or, for example, a power law times a logarithm. A more detailed study of the fixed points reveals that slowly varying multiplicative corrections to the leading power law are generic. Indeed, pure power law behaviour occurs only if the disorder is either of class a or of class c and the probability distribution itself has a pure power law tail. This contrasts to most other applications of the RG method where logarithmic corrections are found only at the critical dimensions. We have not explicitly treated the upper critical dimension, $d_c = 2$. We expect, as in other critical phenomena, that the exponents will be continuous at d_c and that additional slowly varying corrections will appear.

For self similar lattices such as percolation clusters or Sierpinski gaskets the fractal dimension L^{d_f} is defined as the exponent relating the number of lattice sites in a region to its linear dimension. For these and other recently studied fractal structures, each lattice site has the same capacity. However, for site disordered lattices, the sites are not equivalent; sites with large values of τ accommodate more walkers than those with small values of τ. It seems reasonable then to define the fractal dimension for site disordered lattices as the exponent relating the sum of the τ's within a region to the linear dimension of the region. This definition implies that the equilibrium number of walkers within a region of radius L will be proportional to L^{d_f}. Alternatively, in the electrical network picture, the total capacitance of a region is proportional to L^{d_f} while in the harmonic lattice picture the total mass within a region is proportional to L^{d_f}.

Care is required in implementing this definition for class b and c distributions where μ fails to exist. We define the fractal dimension of a site disorded lattice via

$$\left\langle \left(\sum_{j \in B(L)} \tau_j \right)^{\varepsilon} \right\rangle^{1/\varepsilon} \sim L^{d_f} \qquad (30)$$

where $B(L)$ is the set of lattice points within a ball of radius L and where $\varepsilon = 1$ for class a and $0 < \varepsilon < \alpha$ for classes b and c. Although μ may diverge, the average in (30) is guaranteed to exist. It follows from (19)-(21) that for weak disorder, $d_f = d$ while for strong disorder,

$$d_f = d/\alpha. \qquad (31)$$

In contrast to fractal structures studied hitherto, the fractal dimension of site disordered lattices exceeds the Euclidean dimension and can be continuously varied between d and ∞ as α is varied from 1 to 0. One finds the following inequality amongst the dimensions calculated here

$$0 \leqslant d_s \leqslant d \leqslant d_f \qquad (32)$$

whereas for ordinary lattices all three dimensions coincide.

Combining (26) and (31) yields a hyperscaling relation, valid for $d < d_c$, between the fractal dimension of site disordered lattices and the asymptotics of random walks on these lattices,

$$1/\nu = 2 + d_f - d. \qquad (33)$$

The picture which emerges from the RG analysis is analogous to the situation for equilibrium critical phenomena when R is identified with the correlation length and $1/t$ with the reduced temperature. Above the upper critical dimension a mean field theory holds and the correlation length exponent is independent of dimension. The mean field theory is the CTRW approximation for disordered hopping models developed by Scher and Lax (1973), Scher and Montroll (1975) and Klafter and Silbey (1980). For $d < d_c$, the fixed points are described by stable probability distributions and the exponents depend on the fractal dimension and the Euclidean dimension of the lattice. The technique used here reveals the close connection, first discussed by Jona-Lasinio (1975), between the theory of stable probability distributions and the renormalisation group methodology.

I would like to thank R A Guyer and H Gould for useful discussions. I am indebted to D Fisher and S Alexander for pointing out an error in an early version of the paper. This work was supported in part by the National Science Foundation (DMR-7909592 and DMR-8317442).

References

Alexander S 1981 *Phys. Rev.* B **23** 2951
Alexander S, Bernasconi J, Schneider W R and Orbach R 1981 *Rev. Mod. Phys.* **53** 175
de Gennes P G 1976 *Recherche* **7** 919
Feller W 1971 *An introduction to probability theory and its applications* vol 2, 2nd edn (New York: Wiley)
Fisher D S 1984 *Phys. Rev.* A **30** 960
Gefen Y, Aharony A, Mandelbrot B B and Kirkpatrick S 1981 *Phys. Rev. Lett.* **47** 1771
Gillis J E and Weiss G H 1970 *J. Math. Phys.* **11** 1307
Haus J W, Kehr K W and Lyklema J W 1982 *Phys. Rev.* B **25** 2905
Jona-Lasinio G 1975 *Nuovo Cimento* **26B** 99
Klafter J and Silbey R 1980 *J. Chem. Phys.* **72** 843
Luck J M 1983 *Nucl. Phys.* B **225** 169
Machta J 1981 *Phys. Rev.* B **24** 5260
—— 1983 *J. Stat. Phys.* **30** 305
Mandelbrot B B 1982 *Fractal Geometry of Nature* (San Francisco: Freeman)
Nieuwenhuizen Th M and Ernst M H 1985 *Phys. Rev.* B to appear
Scher H and Lax M 1973 *Phys. Rev.* B **7** 4491
Scher H and Montroll E 1975 *Phys. Rev.* B **12** 2455
Schlesinger M F 1974 *J. Stat. Phys.* **10** 421
Sinai Ya G 1982 *Theory Prob. Appl.* **27** 246
Stephen M J and Kariotis R 1982 *Phys. Rev.* B **26** 2917